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MAXIMUM LIKELIHOOD ESTIMATION OF THE POSITION OF A RADIATING SO--ETC(U)  
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MAXIMUM LIKELIHOOD ESTIMATION OF THE POSITION OF A  
RADIATING SOURCE IN A CLOSED WAVEGUIDE

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## ABSTRACT

An array of sensors is receiving radiation from a source of interest. The source and the array are in a closed three dimensional waveguide. The maximum-likelihood estimators of the coordinates of the source are analysed under the assumptions that the noise field is Gaussian, and the source lies on a finite linear manifold spanned by a set of orthogonal functions. These functions are the eigenfunctions of the wave operator given the boundary conditions defining the waveguide, and are called normal modes. The Cramer-Rao lower bound is of the order of the number of modes which define the source excitation function.

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Maximum Likelihood Estimation of the Position of a  
Radiating Source in a Closed Waveguide

Melvin J. Hinich

Introduction

Arrays of interconnected receivers are used to improve the signal-to-noise ratio of coherent radiation from a source located in the same waveguide as the array. A waveguide is a physical science term for the classical wave operator and its associated boundary conditions. In most examples, the phases of the received signals are adjusted by digital or analog methods in order to concentrate the received energy in a narrow beam with respect to the coordinate system of the array. This procedure (called beamforming) is used to estimate the direction of a point source which is radiating coherent energy at a single frequency or in a band of frequencies. Beamforming is widely used for processing arrays in radio astronomy, underwater acoustics, phased array radars, seismology, and atmospheric physics. The statistical properties of beamforming has been studied by Levin [11], Clay, Hinich, and Shaman [6], and Green, Kelley, and Levin [7].

Beamforming is an essential element of the maximum likelihood estimator of source direction if the received radiation is a plane or cylindrical wave. Beamforming will result in a biased or very imprecise estimate of direction if the received wave fronts are significantly effected by reflections from the boundaries of the waveguide, or are convoluted by refraction and dispersion in the medium of propagation (see Clay [4]). Such is the case when the problem is estimating the depth of a source in an infinite

stratified horizontal waveguide. Clay [5] reviews signal processing theory for such a waveguide using both horizontal and vertical arrays, and relates the statistical models to physical oceanography. The maximum likelihood estimator of source direction using a horizontal array is given by Capon et. al [3], and Hinich and Shaman [8]. The maximum likelihood estimator of source depth using a vertical array is given by Hinich [9], [10].

This paper considers the problem of estimating the coordinates of a source in a closed three dimensional waveguide when the source excites a finite number of eigenfunctions of the wave operator. The closure of the environment results in a fundamental limitation of the precision the estimator irrespective of the number of receivers used in the array. This limitation is due to the limited dimensionality of the model which is induced by the physics of the system.

### 1. Normal Mode Solution of the Wave Equation

The steady-state solution of wave equation for an infinite horizontal waveguide is a special case of the general solution of the inhomogeneous Sturm-Liouville partial differential operator on the space  $L_2(S)$  of square integrable functions on  $x \in S$ , a smoothly closed and bounded subset of Euclidean space (Chapter 5, Vladimirov [14]). For expository purposes we first limit attention to one space dimension, and set  $S = [0,1]$ . For  $x$  in the unit interval  $[0,1]$ , let  $f(x)$  denote the twice continuously differentiable solution to the inhomogeneous

Sturm-Liouville problem  $Lf = s(x-x_0)$ , where the operator is defined by

$$Lf = -\frac{\partial}{\partial x} p \frac{\partial}{\partial x} f + qf, \quad (1)$$

$s(x-x_0)$  is the intensity of a source at position  $x_0 \in S$ ,  $p$  is a given continuously differentiable positive function of  $x \in S$ ,  $q$  is non-negative and continuous in  $[0,1]$ , and  $f$  satisfies the following boundary conditions given  $c_i \geq 0$ ,  $i = 1, \dots, 4$ :

$$\begin{aligned} c_1 f(0) - c_2 \frac{\partial}{\partial x} f(0) &= 0 \\ c_3 f(1) + c_4 \frac{\partial}{\partial x} f(1) &= 0. \end{aligned} \quad (2)$$

For steady-state processes  $f(x)$  is the solution to a general wave equation (or a diffusion equation) with the given boundary conditions with normalized space units. The time component in the solution is separable from the space component.

The Sturm-Liouville operator  $L$  is self-adjoint, its eigenvalues  $\gamma_1 \leq \gamma_2 \leq \dots$  are discrete and non-negative, and its eigenfunctions  $\phi_1, \phi_2, \dots$  are complete in  $L_2(S)$ . As an example, consider the simplest model of the ocean as an acoustic waveguide, the homogeneous compressible fluid waveguide with a rigid bottom and a free surface. In the absence of gravity effects, the eigenfunctions for this waveguide are  $\phi_m = \sqrt{2} \cos \gamma_m x$ , and the eigenvalues are  $\gamma_m = (m + 1/2)\pi$ ,  $m = 1, 2, \dots$ , where the units have been normalized so that the depth of the guide is one unit.

In the general model, assume that  $\gamma_1 > 0$  ( $L$  is non-singular).

Then

$$f(x) = \sum_{m=1}^{\infty} (s, \phi_m) \gamma_m^{-1} \phi_m(x) \quad (3)$$

where

$$(s, \phi_m) = \int_0^1 s(x) \phi_m(x) dx$$

is the inner product between  $s$  and  $\phi_m$ . Moreover the eigenfunctions are orthonormal, i.e.  $(\phi_m, \phi_m) = 1$  and  $(\phi_m, \phi_{m'}) = 0$  for  $m \neq m'$ .

When  $L$  is the wave operator, the  $\phi_m$  are called the normal modes.

Suppose that the source excites only the first  $M$  eigenfunctions, i.e. the source function lies on the linear manifold spanned by  $\phi_1, \dots, \phi_M$ . The source function then can be written

$$s(x-x_o) = \sum_{m=1}^M \theta_m(x_o) \phi_m(x), \quad (4)$$

where  $\theta_1, \dots, \theta_M$  are unknown weights which are twice continuously differentiable functions of the source position  $x_o$ . For any  $s \in L_2(S)$ , the function can be approximated to any preassigned tolerance by choosing  $M$  sufficiently large. As long as  $M$  is finite, it is clear from (4) that the source's energy does not emanate from a single point, i.e. in physics terminology the source is not a point source. By the orthogonality of the eigenfunctions, it follows from (3) and (4) that

$$f(x|x_o) = \sum_{m=1}^M \theta_m(x_o) \gamma_m^{-1} \phi_m(x). \quad (5)$$

Thus the received signal also lies on the  $\phi_1, \dots, \phi_M$  linear manifold.

Expressions (4) and (5) holds with  $\underline{x} \in S$  for  $S \subset \mathbb{R}^3$ , and a source which excites only the first  $M$  normal modes. The eigenfunctions  $\phi_m$  satisfy the boundary conditions

$$\alpha(\underline{x})f(\underline{x}) + \beta(\underline{x}) \frac{\partial f}{\partial \underline{n}} = 0, \quad (6)$$

where  $\underline{n}$  is the external normal vector to the (piecewise smooth) boundary of  $S$ , and  $\alpha, \beta \geq 0$  are continuous on the boundary with  $\alpha + \beta > 0$ .

Moreover the operator is

$$L = -\operatorname{div}(p\nabla) + q, \quad (7)$$

where  $\nabla$  is the gradient operator and  $\operatorname{div}$  is the divergence. Estimating the source position  $\underline{x}_0$  involves a linear statistical model whose structure is determined by the physics of the waveguide. The eigenfunctions of the operator  $L$  constitute a "natural" basis for the linear design.

## 2. A Statistical Model

Assume that the source and the receiving array is in a closed three dimensional waveguide as modeled by expressions (6) and (7). The finiteness of the waveguide shapes the structure of the noise field if the source excites only the first  $M$  normal modes.

Let  $\underline{x}_1, \dots, \underline{x}_n$  denote the positions of the receivers in the array, and as before let  $\underline{x}_0 \in S$  denote the position of the source of interest. The coordinates  $(x_{01}, x_{02}, x_{03})$  of  $\underline{x}_0$  are the unknown parameters which are to be estimated from a sample of the signals received by the array.

Suppose that the signal has phase coherent energy in a narrow band about frequency  $f_0$ . Let  $y(\underline{x}_i)$  denote the output from the  $i$ th receiver

in the array which has been filtered in a narrow band about  $f_0$ . For a discussion of filtering in mathematical statistical terminology, see Anderson [1] and Brillinger [2]. If the signal has energy in many frequency bands, the results in this paper apply to each narrowband component of the received signal.

By the linearity of the wave operator,

$$y(\underline{x}_i) = f(\underline{x}_i | \underline{x}_0) + \epsilon(\underline{x}_i),$$

where  $\epsilon(\underline{x}_i)$  is the filtered signal received at  $\underline{x}_i$  from the ensemble of noise sources in S. Assuming that the phases of the noise field in the narrow band around  $\omega$  are incoherent, the signal-to-noise ratio is increased by filtering the received signal in the  $f_0$  band.

Assume that each noise source excites at most N modes, where  $N > > M$ . It then follows from (4) and (5) that  $\epsilon(\underline{x}_i)$  lies on the linear manifold spanned by  $\phi_1, \dots, \phi_N$ . Let  $\epsilon_m$  replace  $\theta_m$  in the linear combination relating  $\epsilon(\underline{x})$  to the  $\phi_m(\underline{x})$ .

In order to simplify the stochastic structure of the noise field  $\epsilon(\underline{x})$ , assume that

$$\epsilon(\underline{x}) = \sum_{m=1}^N \epsilon_m \gamma_m^{-1} \phi_m(\underline{x}), \quad (8)$$

where  $\epsilon_1, \dots, \epsilon_N$  are uncorrelated Gaussian random variables with zero mean and variances  $E\epsilon_m^2 = \omega_m^2$ . Thus the variance-covariance matrix of the noise observed at n locations  $\underline{x}_1, \dots, \underline{x}_n$  is the  $n \times n$  matrix

$$\Sigma_x = \underline{\phi}_N \Gamma_N^{-1} \Omega_N \Gamma_N^{-1} \underline{\phi}^T, \quad (9)$$

where  $\underline{\Phi}$  is the  $n \times N$  matrix whose  $(i, m)$  element is  $\phi_m(x_i)$ ,  $\Gamma_N$  is the  $N \times N$  diagonal matrix with elements  $\gamma_m$ , and  $\Omega_N$  is the  $N \times N$  diagonal matrix with elements  $\omega_m^2$  ( $m = 1, \dots, N$ ).

The matrix  $\underline{\Sigma}_x$  is positive semi-definite with rank at most  $N$  if  $n \geq N$ . As can be seen from expression (8), there are only  $N$  independent realizations from the stochastic noise process during the sampling period regardless of the number of sensors in the array. Consequently there is no reason to have  $n$  greater than  $N$ . In practice, the number of modes needed to provide a good approximation to the noise field must be experimentally determined, as is the case for the source radiation.

Setting the number of sensors equal to  $N$ , suppose that the sensor locations are chosen so that  $\underline{\Phi}$  is non-singular. Thus,

$$\underline{\Sigma}_x^{-1} = (\underline{\Phi}^{-1})^T \underline{\Gamma}_{\Omega_N}^{-1} \underline{\Gamma}_{\Phi}^{-1} \quad (10)$$

The Fisher information is the positive definite  $3 \times 3$  matrix

$$\underline{I}_x(x_o) = (\nabla f_x)^T \underline{\Sigma}_x^{-1} (\nabla f_x) \quad (11)$$

where  $\nabla f_x$  is the  $N \times 3$  matrix whose  $i$ th row in the gradient  $(\frac{\partial}{\partial x_{o1}} f(x_i | x_o), \frac{\partial}{\partial x_{o2}} f(x_i | x_o), \frac{\partial}{\partial x_{o3}} f(x_i | x_o))$  at  $x_o$ . Since  $f(x | x_o) = \sum_{m=1}^M \theta_m(x_o) \gamma_m^{-1} \phi_m(x)$ , then from (10) and (11) we have for all  $N \geq M$ ,

$$\underline{I}_x(x_o) = [\nabla \theta(x_o)]^T \underline{\Omega}_M^{-1} [\nabla \theta(x_o)] \quad (12)$$

where  $\underline{\Omega}_M$  is the diagonal matrix with elements  $\omega_1^2, \dots, \omega_M^2$  and  $\nabla \theta(x_o)$  is the  $M \times 3$  matrix whose  $m, j$ th element is  $\frac{\partial}{\partial x_{oj}} \theta_m(x_o)$ .

It is clear from (12) that the elements of  $\mathbf{I}_{\mathbf{x}}$  are independent of the array size as long as  $n \geq M$ . This means that only M sensors are needed for the maximum likelihood estimation of  $\mathbf{x}_0$ , provided that the  $\mathbf{x}_i$ 's are chosen to make the rows of  $\Phi$  linearly independent vectors. This result still holds when the variance-covariance matrix of the  $\epsilon_m$  is not diagonal, i.e. the  $\epsilon_m$  are correlated. In this case the  $\Omega_M$  matrix in (12) is the  $M \times M$  variance-covariance of  $(\epsilon_1, \dots, \epsilon_M)$ .

For each  $j = 1, 2, 3$  and  $m = 1, \dots, M$ , let  $B_{mj}$  denote the maximum of  $|\frac{\partial}{\partial \mathbf{x}_{0j}} \theta_m(\mathbf{x}_0)|$  for  $\mathbf{x}_0 \in S$ . Given a vector  $\mathbf{a} = (a_1, a_2, a_3)^T$  of unit length, it follows from (11) by the Schwarz inequality that

$$\begin{aligned} \mathbf{a}^T \mathbf{I}_{\mathbf{x}}(\mathbf{x}_0) \mathbf{a} &\leq \sum_{m=1}^M \omega_m^{-2} (B_{m1}^2 + B_{m2}^2 + B_{m3}^2) \\ &\leq 3\lambda^{-1} \bar{B}_M^2, \end{aligned} \tag{13}$$

where  $\bar{B}^2 = \max_{m,j} \{B_{mj}\}$  and  $\lambda = \min_m \{\omega_m^2\}$ . Thus the variance of an unbiased estimator of  $\mathbf{x}_{0j}$  is bounded below by  $\lambda(3\bar{B}_M^2)^{-1}$ . For a non-diagonal  $\Omega_M$ ,  $\lambda$  is replaced by the minimum eigenvalue of  $\Omega_M$ .

### 3. Maximum Likelihood Estimation of $\mathbf{x}_0$

Assume that the  $M$  sensor array is constructed so that the  $M \times M$  matrix  $\Phi = (\phi_m(\mathbf{x}_1))$  is non-singular. Then the model can be written

$$\tilde{\mathbf{y}} = \tilde{\mathbf{\Phi}}^{-1} (\theta(\mathbf{x}_0) + \tilde{\boldsymbol{\epsilon}}), \tag{14}$$

where  $\tilde{\mathbf{y}} = (y(\mathbf{x}_1), \dots, y(\mathbf{x}_M))^T$ ,  $\theta(\mathbf{x}_0) = (\theta_1(\mathbf{x}_0), \dots, \theta_M(\mathbf{x}_0))^T$ , and  $\tilde{\boldsymbol{\epsilon}} = (\epsilon(\mathbf{x}_1), \dots, \epsilon(\mathbf{x}_M))^T$ . Since the  $\epsilon(\mathbf{x}_i)$  are normally distributed with variance-covariance matrix  $\Omega_M$ , then the first order conditions for

the maximum likelihood estimator of  $\tilde{x}_o$  are

$$(\Gamma \Phi^{-1} \tilde{y} - \theta(\tilde{x}_o))^T \Omega_M^{-1} [\nabla \theta(\tilde{x}_o)] = 0, \quad (15)$$

where  $\nabla \theta(\tilde{x}_o)$  is the  $M \times 3$  matrix  $(\frac{\partial}{\partial \tilde{x}_{oi}} \theta_m(\tilde{x}_o))$ . The maximum likelihood estimator  $\hat{\tilde{x}}_o$  is one of the solutions to equation (15) (see Chapter 5, Rao [12]).

It has been assumed that  $\theta(\tilde{x}_o)$  is twice continuously differentiable. In addition, assume that  $|\frac{\partial^3}{\partial \tilde{x}_{oi} \partial \tilde{x}_{oj} \partial \tilde{x}_{ok}} \theta(\tilde{x}_o)|$  has a finite upper bound for all  $\tilde{x}_o \in S$  and  $i, j, k = 1, 2, \text{ or } 3$ . Given the normality of the errors in the model, it follows that as  $M \rightarrow \infty$ ,  $M^{\frac{1}{2}} (\hat{\tilde{x}}_o - \tilde{x}_o)$  is asymptotically normally distributed  $N(0, I_{\tilde{x}_o}^{-1}(\tilde{x}_o))$ .

#### 4. Nonstationarity of the Waveguide

It is often the case that the medium or the boundaries of the waveguide are slightly nonstationary during the period when the received signals are being filtered. In order to model the effect of this nonstationarity, assume that the filtered signal received at  $\tilde{x}_i$  from a source at  $\tilde{x}_o$  is given by the expression

$$f(\tilde{x}_i | \tilde{x}_o)' = \sum_{m=1}^M \theta_m(\tilde{x}_o) [\gamma_m^{-1} \phi_m(\tilde{x}_i) + u_m(\tilde{x}_i)], \quad (16)$$

where  $E u(\tilde{x}_i) = 0$ ,  $E u_m^2(\tilde{x}_i) = \sigma_m^2$ , and  $u(\tilde{x}_i)$  is independent of  $\varepsilon(\tilde{x}_i)$  for each  $i = 1, \dots, M$ . In other words, expression (5) is modified by adding a stochastic perturbation term  $u(\tilde{x}_i)$  to  $\gamma_m^{-1} \phi_m$  (see Chapter 6, Tolstoy and Clay [13]). The results given in the previous sections still hold as long as the  $\theta_m(\tilde{x}_o)$  have the previously assumed smoothness, but  $\Omega_M$  is replaced by  $\Omega_M + \Sigma_u$ , where  $\Sigma_u$  is the variance-covariance

matrix of the perturbation terms. Assume that  $\Sigma_u$  is non-singular and is independent of the array geometry. The perturbation effect raises the asymptotic variances of the maximum likelihood estimators of the  $x_{oj}$ , since

$$\underline{a}^T [\nabla \theta(\underline{x}_0)]^T \underline{\Omega}_M^{-1} [\nabla \theta(\underline{x}_0)] \underline{a} < \underline{a}^T [\nabla \theta(\underline{x}_0)]^T (\underline{\Omega}_M + \underline{\Sigma}_u)^{-1} [\nabla \theta(\underline{x}_0)] \underline{a} \quad (17)$$

for  $\underline{a} \neq 0$  and  $\nabla \theta \neq 0$ . This result is hardly surprising. It is also clear that if  $\sigma_m^2$  is very large for large  $m$ , the precision of the maximum likelihood estimator is limited as if  $M$  was bounded.

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